ML4Q - Quantum Technologies - Tutorial Solutions Sheets 1 and 2

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March 2020

Sheet 1: Density matrix, Bloch states and Controlled Unitaries

1.1

We are given the state

$$|\psi\rangle = e^{i\phi}\big(\cos\frac{\theta}{2}|0\rangle + e^{i\alpha}\sin\frac{\theta}{2}|1\rangle\big)\,.$$

a)

Construct the according density matrix via $\rho = |\psi\rangle\langle\psi|$.
First the global phase $e^{i\phi}$ cancels due to adjoint $\langle\psi|$.

We obtain

$$\begin{aligned}\rho_\psi = |\psi\rangle\langle\psi| &= \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\alpha}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\alpha}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2}\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \cos\theta + 1 & \sin\theta(\cos\alpha - i\sin\alpha) \\ \sin\theta(\cos\alpha + i\sin\alpha) & 1-\cos\theta\end{pmatrix} \\ &= \frac{\mathbb{I} + \cos\theta\,\sigma_z + \sin\theta\cos\alpha\,\sigma_x + \sin\theta\sin\alpha\,\sigma_y}{2}\,.\end{aligned}$$

such that the coefficients are $n_x = \sin\theta\cos\alpha$, $n_y = \sin\theta\sin\alpha$ and $n_z = \cos\theta$. Using $\sin^2\phi + \cos^2\phi = 1$ it is easy to see that $\|n\| \leq 1$. In fact, even the equality holds, due to $|\psi\rangle$ being a pure state, more on that later.

b)

We can represent any mixed state ρ_M as a sum of pure states $|\psi_i\rangle$, such that

$$\rho_M = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i p_i \rho_i$$

Using the linearity of the sum, we write

$$\rho_M = \sum_i p_i \frac{1}{2}\Big(\mathbb{I} + n_{x_i}\sigma_x + n_{y_i}\sigma_y + n_{z_i}\sigma_z\Big) \tag{1}$$

and just redefine the coefficients as $\tilde{n}_k = \sum_i p_i n_{k_i}$ for $k \in \{x, y, z\}$ and obtain again a valid description in the Bloch basis.

c)

We obtain $\rho = \frac{\mathbb{I}}{2}$ from $n_k = 0$ for $k \in \{x, y, z\}$. This refers to the maximally mixed state. (Can be checked via some measure of entropy)

d)

Show that ρ is pure iff $\|n\| = 1$

There are numerous (but mathematically equivalent) properties of a density matrix ρ, that show whether or not ρ refers to a mixed or a pure state. Here are some

(*i*) $\rho = \rho^2$

(*ii*) $\mathrm{Tr}[\rho^2] = 1$

(*iii*) $\mathrm{rank}\rho = 1$

(*iv*) ...

Here we will use the first one, i.e. it remains to show that $\rho = \rho^2$ iff $\|n\| = 1$.

Compute ρ^2 for

$$\rho = \frac{\mathbb{I} + n_x\sigma_x + n_y\sigma_y + n_z\sigma_z}{2}$$

This yields 16 terms, which are the following

- 6 terms: all terms between the $\mathbb{I}$ term and the σ-terms giving $\rho - \frac{1}{2}\mathbb{I}$
- 6 terms between the σ-terms. These vanish due to anti-commutation $[\sigma_i, \sigma_j]_+ = \delta_{ij} 2\,\mathbb{I}$
- the 4 terms squared. These give $\frac{1}{4}(1 + \|n\|^2)\mathbb{I}$.

Hence, summing all of these we will obtain $\rho = \rho^2$ iff $\|n\| = 1$.

1.2

We consider the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle - \alpha|1\rangle \ ,$$

the Pauli matrices and the Hadamard gate

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

a)

We require $\langle\psi|\psi\rangle = 1$, i.e. $1 = \frac{1}{3} + \alpha \cdot \alpha^*$. This equation is satisfied for $\alpha = e^{i\phi}\sqrt{\frac{2}{3}}$, therefore we cannot define α uniquely, the phase remains arbitrary.

b)

$\alpha \in \mathbb{R}_+$ iff $\phi = 0$, hence the state is $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle$. Action of said operators gives

- $\sigma_x|\psi\rangle = -\sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle$
- $\sigma_y|\psi\rangle = i\sqrt{\frac{2}{3}}|0\rangle + i\frac{1}{\sqrt{3}}|1\rangle$
- $\sigma_z|\psi\rangle = \sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$
- $H|\psi\rangle = \frac{1}{\sqrt{6}}\big((1-\sqrt{2})|0\rangle + (1+\sqrt{2})|1\rangle\big)$

c)

Acting with the Hadamard gate on the general density matrix ρ is given by $H\rho H$. Such an action switches coefficients to

$$H\rho H = \frac{\mathbb{I} + n_z\sigma_x + n_x\sigma_z - n_y\sigma_y}{2} \ .$$

This will not change $\|n\|$ and the purity of the state. This makes sense as local operations can never change the correlation with another system, i.e. the entanglement with another system.

d)

When measuring in the σ_y basis we obtain either one of two eigenstates, which are given by

$$|+_y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \qquad |-_y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

the respective operators then are $|+_y\rangle\langle +_y|$ and $|-_y\rangle\langle -_y|$, such that we can obtain the probability for each of them via

$$\mathbb{P}(+|y) = \mathrm{Tr}[|+\rangle\langle+|_y \ |\psi\rangle\langle\psi|] = |\langle\psi|+_y\rangle|^2 \, .$$

Plugging in our state in this expression we get

$$\langle\psi|+_y\rangle = \frac{1}{\sqrt{2}}\Big(\sqrt{\frac{1}{3}} + i\sqrt{\frac{2}{3}}\Big) \, .$$

Of which we take the squared absolute value $\Rightarrow \mathbb{P}(+|y) = \frac{1}{2}$.

1.3

a)

Define the Pauli group for n qubits as

$$P_n = \left\{ e^{i\theta\pi/2}\sigma_{j_1} \otimes \cdots \otimes \sigma_{j_n} | \theta = 0,1,2,3, j_k = 0,1,2,3 \right\} \, . \tag{2}$$

We want to show that if we act on any Pauli matrix with one of said matrices we remain in the Pauli group.

Action on $\sigma_0 = \mathbb{I}_2$:
All gates are unitary, hence by definition: $U\mathbb{I}U^\dagger = \mathbb{I}$

Action on $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$:

For the phase gate S:

$$S\sigma_z S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For Hadamard gate H:

$$HZH^\dagger = HZH = H \begin{pmatrix} |0\rangle\langle 0| & 0 \\ 0 & -|1\rangle\langle 1| \end{pmatrix} H = \begin{pmatrix} |+\rangle\langle +| & 0 \\ 0 & -|-\rangle\langle -| \end{pmatrix} = X$$

For CNOT gate:

$$\begin{aligned} CNOT_{12} Z_2 CNOT_{12} &= H_2 CZ_{12} H_2 Z_2 H_2 CZ_{12} H_2 = H_2 CZ_{12} X_2 CZ_{12} H_2 \\ &= H_2 \big(|0\rangle\langle 0|_1 \otimes \mathbb{I}_2 + |1\rangle\langle 1|_1 \otimes Z_2 \big) X_2 CZ_{12} H_2 = H_2 X_2 \big(|0\rangle\langle 0|_1 \otimes \mathbb{I}_2 - |1\rangle\langle 1| \otimes Z_2 \big) CZ_{12} H_2 \\ &= H_2 X_2 Z_1 CZ_{12} CZ_{12} H_2 = Z_1 \otimes Z_2 \end{aligned}$$

Action on $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$:

For S gates:

$$SXS^\dagger = Y \quad \Rightarrow \quad S^\dagger Y S = X$$

For H gates: (us previous result)

$$HXH = Z$$

For CNOT gates:

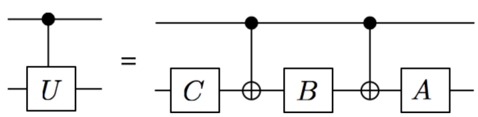
$$CNOT_{12}X_2CNOT_{12} = H_2CZ_{12}H_2X_2H_2CZ_{12}H_2 =$$
$$= H_2CZ_{12}Z_2CZ_{12}H_2 = \mathbb{I}_1 \otimes X_2$$

b)

We are interested in a general single-qubit unitary U given by

$$U = R_z(\beta)\, R_y(\gamma)\, R_z(\delta)\ ,$$

where for $k \in \{x, y, z\}$ the given functions are $R_k(\theta) = \exp(-i\frac{\theta}{2}\sigma_k)$.
Our goal is to show that a two-qubit controlled-U gate can be built from a combination of CNOT gates and single-qubit rotations, according to the quantum circuit construction below:



Here, A, B and C are single-qubit gates, chosen as the following combinations of single-qubit rotations,

$$A = R_z(\beta)R_y(\gamma/2), \tag{3}$$
$$B = R_y(-\gamma/2)R_z(-(\delta+\beta)/2), \tag{4}$$
$$C = R_z((\delta-\beta)/2). \tag{5}$$

To show that figure creates the desired action, we split the task into three parts.

(i) Show that $ABC = \mathbb{I}$ holds.
This is done by using commutation between equivalent operators $[\sigma_k, \sigma_k] = 0$.

$$ABC = e^{-i\frac{\beta}{2}\sigma_z}\, e^{-i\frac{\gamma}{4}\sigma_y}\, e^{i\frac{\gamma}{4}\sigma_y}\, e^{i\frac{(\delta+\beta)}{4}\sigma_z}\, e^{-i\frac{(\delta-\beta)}{4}\sigma_z} = e^{-i\frac{\beta}{2}\sigma_z}\, e^{i\frac{\beta}{2}\sigma_z} = \mathbb{I}$$

(i) Show that $XR_y(\theta)X = R_y(-\theta)$.

$$Xe^{-i\frac{\theta}{2}\sigma_y}X = X\big(\cos\frac{\theta}{2}\,\mathbb{I} - i\sin\frac{\theta}{2}\,\sigma_y\big)X = R_y(-\theta) \tag{6}$$

Also remember $XZX = -Z$

Use this equality to calculate $AXBXC$:
We notice that XBX is

$$XBX = XR_y(-\frac{\gamma}{2})R_z(-\frac{(\delta+\beta)}{2})X = XR_yXXR_zX = R_y(\frac{\gamma}{2})R_z(\frac{(\delta+\beta)}{2}) \tag{7}$$

Plugging this in we obtain $AXBXC = U$. Now we also consider also the controlled operation, giving us

$$A_2\big(\left|0\middle\rangle\!\middle\langle 0\right| \otimes \mathbb{I}_2 + \left|1\middle\rangle\!\middle\langle 1\right| \otimes X_2\big)B_2\big(\left|0\middle\rangle\!\middle\langle 0\right| \otimes \mathbb{I} + \left|1\middle\rangle\!\middle\langle 1\right| \otimes X_2\big)C_2 =$$
$$= |0\rangle\langle 0| \otimes A_2B_2C_2 + |1\rangle\langle 1| \otimes A_2X_2B_2X_2C_2 = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes U$$

Sheet 2: Entanglement

2.1

a)

Show that the two-qubit pure state is entangled:

$$\left|\Psi^+\right\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

If the state is not entangled, but instead a product state, there exist probability amplitudes such that we can write

$$\left|\Psi^+\right\rangle = (\alpha_1 |0\rangle + \beta_1 |1\rangle) \otimes (\alpha_2 |0\rangle + \beta_2 |1\rangle) \, .$$

To obtain the desired that we require $\alpha_1\alpha_2 = \beta_1\beta_2 = \frac{1}{\sqrt{2}}$ while also $\alpha_1\beta_2 = \alpha_2\beta_1 = 0$. Such variables do not exist, therefore the state is not a product state.

b)

Application of the Pauli matrices to the state gives

$$X_1\left|\Psi^+\right\rangle = X_2\left|\Psi^+\right\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$Z_1\left|\Psi^+\right\rangle = Z_2\left|\Psi^+\right\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$Y_1\left|\Psi^+\right\rangle = Y_2\left|\Psi^+\right\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$$

In the last equation we neglected the global phase.
The entanglement of the state remains unchanged as local operations never change entanglement.

c)

We show the (up to a local unitary) equivalence between the state $|\Psi^+\rangle$ and applying the CZ gate to $|+\rangle|+\rangle$. Proof:

$$CZ_{12}|+\rangle \otimes |+\rangle = \big(|0\rangle\langle 0| \otimes \mathbb{I}_2 + |1\rangle\langle 1| \otimes Z\big)|+\rangle \otimes |+\rangle =$$
$$= \frac{1}{\sqrt{2}}\big(|+\rangle|0\rangle + |-\rangle|1\rangle\big)$$

Applying the Hadamard gate to the first qubit gives the desired state $|\Psi^+\rangle$.

d)

We evaluate the mixture of multiple entangled states using the density matrix formalism, such that we obtain

$$\rho = \frac{1}{2}|\Psi^+\rangle\langle\Psi^+| + \frac{1}{2}Z_2|\Psi^+\rangle\langle\Psi^+|Z_2 = \frac{1}{4}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1&0&0&-1\\0&0&0&0\\0&0&0&0\\-1&0&0&1\end{pmatrix} =$$

$$= \frac{1}{2}|0\rangle\langle 0|_1 \otimes |0\rangle\langle 0|_2 + \frac{1}{2}|1\rangle\langle 1|_1 \otimes |1\rangle\langle 1|_2$$

which is not an entangled state.

For an unequal mixing according to

$$\rho = p|\Psi^+\rangle\langle\Psi^+| + (1-p)Z_2|\Psi^+\rangle\langle\Psi^+|Z_2 =$$
$$= \begin{pmatrix}\frac{1}{2}&0&0&p-\frac{1}{2}\\0&0&0&0\\0&0&0&0\\p-\frac{1}{2}&0&0&\frac{1}{2}\end{pmatrix}$$

By the PPT criterion, this state is entangled for $p \neq \frac{1}{2}$.