MLQ4 Exercises, Quantum Information Processing

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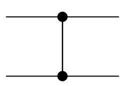
Exercise classes 3 and 4: March 17, 14:15 -16:00

In the first exercise we will talk about a scheme of quantum computation, where measurements are crucial. One of the simplest examples of such schemes are teleportation protocol. Here we derive even simpler scheme and show how Clifford circuit and a special input can boost computation to being universal.

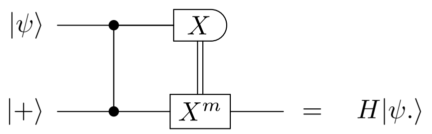
3.1 Magic states and computation

Yesterday we saw that the Bell state can also prepared using a two qubit phase CZ gate. In the circuit model this gate is denotes as follows

CZ Gate

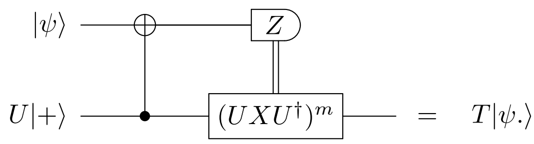


a) Show that the following circuit identity holds for an arbitrary single qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ state, where m is X-measurement outcome.



b) modify the circuit such that it uses a CNOT gate instead of CZ gate and implements an identity operator.

c) The circuit in the previous exercise consists of Clifford gates and measurements: CNOT gate, Pauli, Hadamard gates and a Pauli measurement. As discussed yesterday Clifford gates are not universal. Modify the second input to the circuit by adding a single diagonal unitary operation, such that the circuit can implement a T gate, while the entire circuit including $(UXU^\dagger)^m$ must be a Clifford gate.



Such states $U|+\rangle$ that can boost computation are called *magic* states.

3.2 Quantum Dense coding

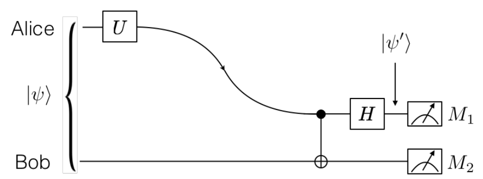
The phenomenon of "dense coding" is based on the observation that given some state belonging to the Bell basis as discussed yesterday there are local unitaries in Alice's lab U_a which will map it onto any of the other states belonging to the basis, apart from an overall phase. There are similar unitaries in Bob's lab U_b

Consider two parties (Alice and Bob) at different spatial locations, initially each of them holding one qubit of a single shared Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{1}$$

Consider the circuit below:

a



b

Alice's U	Bob's meas. outcome	
	M_1	M_2
$\mathbb{I}$		
X		
Z		
ZX		

Alice decides to apply one of four possible gate operations $U = \{\mathbb{I}, X, Z, ZX\}$ to the qubit in her possession and then sends her qubit to Bob, as shown in part a) of the figure. After receiving Alice's qubit, Bob applies a CNOT gate followed by one Hadamard gate to the two qubits, as shown in part a) of the figure above.

a) Recall the two qubit state $|\psi_i\rangle$ for each of the four possible choices for U_i. Now, Bob measures both qubits in the standard computational basis (Z-basis). Determine the corresponding measurement outcomes, i.e. fill in the table of part b).

b) Explain how many bits of classical information this protocol allows Alice to send to Bob.

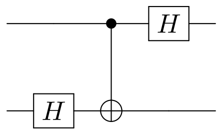
c) Argue how this compares to the maximum amount of classical information that Alice could deterministically send over to Bob (using a different protocol) if she only had access to a single qubit (rather than a qubit that forms part of the Bell state shared with Bob).

d) Explain whether or not the discussed dense-coding protocol allows faster-than-light communication.

3.3 Find a present among four boxes

We are promised that one of the four given boxes contains a present. The boxes are enumerated by $00, 01, 10, 11$. The function $f : \{0,1\}^2 \to \{0,1\}$ gives an output 1 if the box contains a present and 0 otherwise.

Show that we can discover the present with only a single call of this function. Start by initializing a quantum state as $|++-\rangle$ and act on this state using a function f. Then transform the first two qubits using the circuit below and finally we measure them. Show that the results obtained by measurements lets us find out in which box is the preset.



4.1. Quantum error correction: The 3-qubit phase flip code

The 3-qubit phase flip quantum error correcting (QEC) code encodes a single physical qubit,

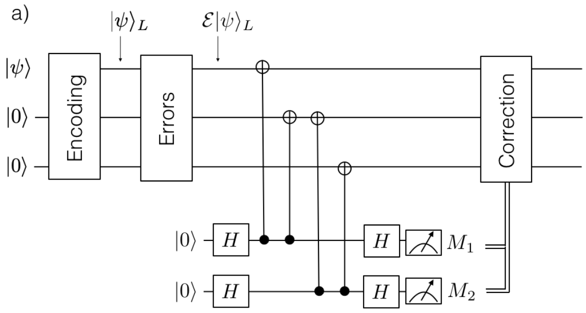
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

in three physical qubits, $|\psi\rangle_L = \alpha|0\rangle_L + \beta|1\rangle_L$. The logical basis states are defined as

$$\begin{aligned} |0\rangle_L &= |+++\rangle, \\ |1\rangle_L &= |---\rangle. \end{aligned}$$

The QEC protocol is shown in the circuit below (part a)) and consists of the encoding step, possible occurrence of phase flip errors, syndrome measurement and correction.

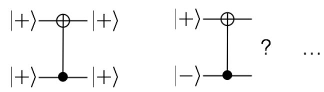
a)



b)

Error	Syndrome		Correction
$\mathbb{I}$	0	0	$\mathbb{I}$
Z_1			
Z_2			
Z_3			

c)



a) Find an encoding circuit, using CNOT and Hadamard gates, which realises the encoding step, $|\psi\rangle|0\rangle|0\rangle \to |\psi\rangle_L$.

Hint: How could the encoding circuit for the 3-qubit bit flip code be modified?

b) Determine the output states (expressed in the $|\pm\rangle$ basis) of a CNOT gate acting on the four possible two-qubit basis states in the X-basis, i.e. the action of the CNOT on $|+\rangle|+\rangle$, $|+\rangle|-\rangle$, $|-\rangle|+\rangle$ and $|-\rangle|-\rangle$ (as shown in subfigure c).

c) Now, consider the error syndrome measurement circuit (central part of a)): here, the three system qubits are coupled via the four CNOT and Hadamard gates to the two ancilla qubits. Subsequently, the two ancilla qubits are both measured in the standard basis (Z-basis).

Show that if no error happens on the register of three system qubits after the encoding, the subsequent syndrome measurement will yield an outcome of $\{M_1, M_2\} = \{0, 0\}$.

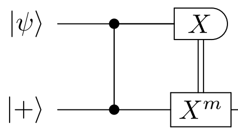
d) Now consider the three cases of a single phase flip error occurring on any of the three qubits, i.e. Z_1, Z_2 or Z_3. Determine the measurement outcomes $\{M_1, M_2\}$ of the syndrome measurement resulting from these error events (middle columns of table b).

e) What correction operations should be applied, depending on the error syndrome $\{M_1, M_2\}$, in order to restore the original encoded state $|\psi\rangle_L$ (last column of table b)?

Solutions

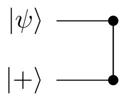
3.1 Magic states and computation

a) We need to verify the following circuit for an arbitrary input $|\psi\rangle$:



$$= \quad H|\psi.\rangle$$

Let us do it step-by-step:



$$= \quad CZ|\psi\rangle|+\rangle$$

We have already done a very similar calculation yesterday:

$$\begin{aligned} |\tilde{\psi}\rangle = CZ|\psi\rangle|+\rangle =& CZ(\alpha|0\rangle + \beta|1\rangle) \otimes (|0\rangle + |1\rangle)/\sqrt{2} \\ =& CZ(\alpha|00\rangle + \alpha|01\rangle + \beta|10\rangle + \beta|11\rangle)/\sqrt{2} \\ =& (\alpha|00\rangle + \alpha|01\rangle + \beta|10\rangle - \beta|11\rangle)/\sqrt{2} \end{aligned}$$

Measure the first qubit in Pauli X basis. Possible outcomes are $|\pm\rangle$:

$$\begin{aligned} \langle \pm_1|\tilde{\psi}\rangle =& \frac{1}{\sqrt{2}}(\langle 0|\tilde{\psi}\rangle) \pm \frac{1}{\sqrt{2}}(\langle 1|\tilde{\psi}\rangle) \\ =& \frac{1}{2}(\alpha|+\rangle \pm \beta|-\rangle) \end{aligned}$$

If the outcome is $-$, X correction is applied. $X|\pm\rangle = \pm|\pm\rangle$. Thus, after correction step we get the final state:

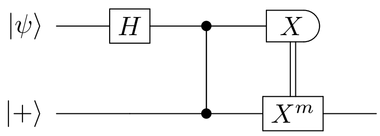
$$\frac{1}{2}(\alpha|+\rangle + \beta|-\rangle)$$

Note that the state is not normalized. This was expected. Recall the rule to derive s post-measurement states and as a result we get

$$\alpha|+\rangle + \beta|-\rangle = H(\alpha|0\rangle + \beta|1\rangle).$$

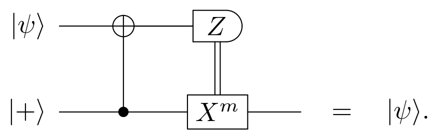
■

b) If we want the circuit to implement identity, we have to use on more Hadamard gate. There is not an unique solution to this question, but since we have a hint to use NOT get we can do the following. Consider the circuit

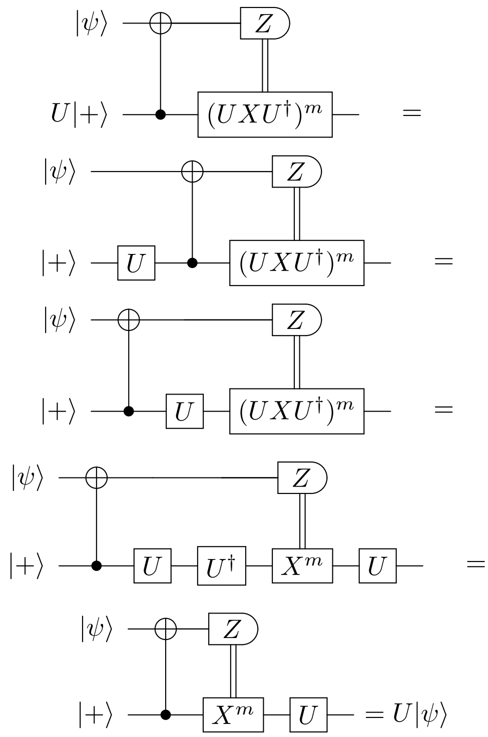


$$= \quad |\psi\rangle.$$

From **a)** we know that this identity holds here. Now let's rewrite it using a CNOT gate.



c) If U is a diagonal unitary, it commutes with $CNOT_{21}$, where controled qubit is 2. Then we have the following gate identity:



Since we want to implement a T gate, let us assign $U = T = diag(1, e^{\frac{i\pi}{4}})$.
Let us check the value of $TXT^\dagger$. One can use 2×2 matrix multiplications to calculate

$$TXT^\dagger = e^{\frac{-i\pi}{4}} YS. \tag{2}$$

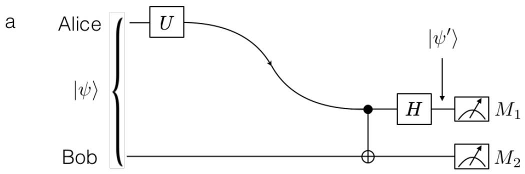
The right hand side is the product of gates in the Clifford group.

To sum up the entire circuit consists only of the Clifford gates and Pauli measurements. You learnt in the class that the Clifford gates alone cannot be used for the universal quantum computation. In addition, it is well-known in quantum information, that having any Clifford circuit and Pauli measurements cannot give any quantum advantage over the classical counterpart. That means that whatever Clifford + Pauli measurement quantum computation can achieve efficiently, the usual classical computer can do too.

Here what changes is that, we keep quantum circuit, but we change the input state. To say it will the popular language, we *inject a magic state*. This magic state is a resource which can be used to boost the Clifford computation to the universal one.

3.2 Quantum Dense coding

a). A single shared Bell state



$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

and Alice's possible unitaries are $U = \{\mathbb{I}, X, Z, ZX\}$. First one is an identity. See the rest below:

$$\begin{aligned}
|\psi_2\rangle =& \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \\
|\psi_3\rangle =& \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
|\psi_4\rangle =& \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).
\end{aligned}$$

In order to get the readout for each measurement, we apply the other gated from the circuit: $|\psi_i'\rangle = H_1 CNOT_{12}|\psi_i\rangle$

$$\begin{aligned}
|\psi_1'\rangle =& \frac{1}{\sqrt{2}}H_1(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|+0\rangle + |-0\rangle) = |00\rangle \\
|\psi_2'\rangle =& \frac{1}{\sqrt{2}}H_1(|11\rangle + |01\rangle) = \frac{1}{\sqrt{2}}(|-1\rangle + |+1\rangle) = |01\rangle \\
|\psi_3'\rangle =& \frac{1}{\sqrt{2}}H_1(|00\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|+0\rangle - |-0\rangle) = |10\rangle \\
|\psi_4'\rangle =& \frac{1}{\sqrt{2}}H_1(|01\rangle - |11\rangle) = \frac{1}{\sqrt{2}}(|+1\rangle - |-1\rangle) = |11\rangle.
\end{aligned}$$

Now we fill in the table:

U	$\lvert\psi'\rangle$	M_1	M_2
$\mathbb{I}$	$\lvert 00\rangle$	0	0
X	$\lvert 01\rangle$	0	1
Z	$\lvert 10\rangle$	1	0
ZX	$\lvert 11\rangle$	1	1

b) There are totally four options to be sent, which can be encoded in four possibilities $\{00, 01, 10, 11\}$. Thus, Alice has to send two bits.

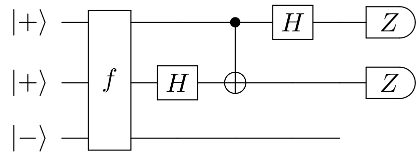
c) If Alice only had access to a single qubit (rather than a qubit that forms part of the Bell state shared with Bob) the maximum amount of classical information that she could deterministically send over to Bob (using a different protocol) would be just a single bit of a classical information: 0

or 1.

d) Faster then speed of light communication is not possible. Alice is space-like separated and after she implements a unitary gate, she has to send her qubit to Bob via quantum channel. This channel itself is of course limited by the speed of light. In this protocol note that no classical information is communicated

3.3 Find a present among four boxes

We show that we can discover the present with only a single call of this function. We start by initializing a quantum state as $|++-\rangle$ and act on this state using a function f, then the circuit and then measurements.



Let us start by the applications of initialization:

$$\frac{1}{2\sqrt{2}} \sum_{x\in\{0,1\}^2} |x\rangle(|0\rangle - |1\rangle).$$

Apply f:

$$\frac{1}{2\sqrt{2}} \sum_{x\in\{0,1\}^2} |x\rangle(|f(x)\rangle - |1 \oplus f(x)\rangle) = \frac{1}{2\sqrt{2}} \sum_{x\in\{0,1\}^2} (-1)^{f(x)}|x\rangle(|0\rangle - |1\rangle).$$

Thus the last qubit remains in a product state $|-\rangle$ and since we are not acting on it any longer, we can forget about it.

Next, we apply Hadamard gate on the second qubit and obtain:

$$\frac{1}{2} \sum_{x\in\{0,1\}^2} (-1)^{f(x)}|x\rangle \longmapsto \frac{1}{2}\Big((-1)^{f(00)}|0+\rangle + (-1)^{f(01)}|0-\rangle + (-1)^{f(10)}|1+\rangle + (-1)^{f(11)}|1-\rangle\Big)$$

Next we apply CNOT_{12}. Remember that CNOT is applying a quantum NOT gate on the cesond qubit, when we have 1 on the first one.

$$\frac{1}{2}\Big((-1)^{f(00)}|0+\rangle + (-1)^{f(01)}|0-\rangle + (-1)^{f(10)}|0+\rangle - (-1)^{f(11)}|1-\rangle\Big)$$

Finally, application of Hadamard on qubit 1, gives:

$$|\psi\rangle = \frac{1}{2}\Big((-1)^{f(00)}|++\rangle + (-1)^{f(01)}|+-\rangle + (-1)^{f(10)}|-+\rangle - (-1)^{f(11)}|--\rangle\Big)$$

We have finally implemented the circuit. Now we have to make measurements. Let us check what we get for all possible measurement outcomes and all possible placements of the present:

1. $f(00) = 1$, thus we get:

$$|\psi\rangle = \frac{1}{2}\Big(-|++\rangle + |+-\rangle + |-+\rangle - |--\rangle\Big) \propto |11\rangle.$$

2. $f(01) = 1$, thus we get:

$$|\psi\rangle = \frac{1}{2}\Big(|++\rangle - |+-\rangle + |-+\rangle - |--\rangle\Big) \propto |01\rangle.$$

3. $f(10) = 1$, thus we get:

$$|\psi\rangle = \frac{1}{2}\Big(|++\rangle + |+-\rangle - |-+\rangle - |--\rangle\Big) \propto |10\rangle.$$

4. $f(11) = 1$, thus we get:

$$|\psi\rangle = \frac{1}{2}\Big(|++\rangle + |+-\rangle + |-+\rangle + |--\rangle\Big) \propto |00\rangle.$$

Thus, with a single call of the function we can find a present.

4.1. Quantum error correction: The 3-qubit phase flip code

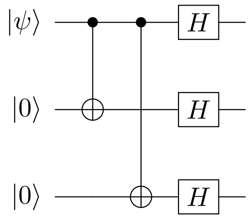
The 3-qubit phase flip quantum error correcting (QEC) code encodes a single physical qubit,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

in three physical qubits, $|\psi\rangle_L = \alpha|0\rangle_L + \beta|1\rangle_L$. The logical basis states are defined as

$$\begin{aligned} |0\rangle_L &= |+++\rangle, \\ |1\rangle_L &= |---\rangle. \end{aligned}$$

a) The following circuit does the encoding $|\psi\rangle|0\rangle|0\rangle \to |\psi\rangle_L$:



One can verify this easily:

$$\begin{aligned} H^{\otimes 3} CNOT_{13} CNOT_{12}(\alpha|000\rangle + \beta|100\rangle) =& H^{\otimes 3}(\alpha|000\rangle + \beta|111\rangle) \\ =& \alpha|+++\rangle + \beta|---\rangle. \end{aligned}$$

b) We know how to handle this, indeed we have seen similar tasks in the previous exercises. A CNOT gate is controlled Pauli X. $X|\pm\rangle = \pm|\pm\rangle$.

We can do this without drawing.

$$CNOT_{21}|+\rangle\frac{(|0\rangle + |1\rangle)}{\sqrt{2}} = |+\rangle|+\rangle.$$

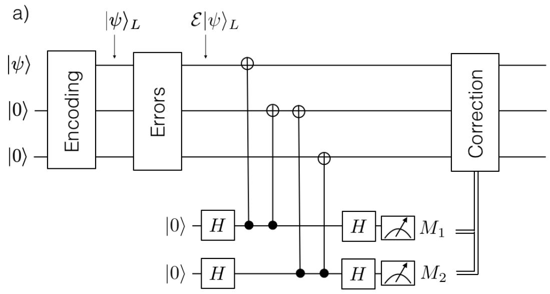
$$CNOT_{21}|-\rangle\frac{(|0\rangle + |1\rangle)}{\sqrt{2}} = |-\rangle|-\rangle.$$

$$CNOT_{21}|+\rangle\frac{(|0\rangle - |1\rangle)}{\sqrt{2}} = |+\rangle|-\rangle.$$

$$CNOT_{21}|-\rangle\frac{(|0\rangle - |1\rangle)}{\sqrt{2}} = |-\rangle|+\rangle.$$

Thus, the second qubit is flipped if the first one is $|-\rangle$.

c) Consider the middle part of the Fig. a). Let us rewrite it here:



We have to consider the action of 4 CNOT gates on $|\phi\rangle_L|++\rangle$. We use part b).

$$\begin{aligned} CNOT_{41}(\alpha|+++\rangle+\beta|---\rangle)|++\rangle =& \alpha|+++\rangle|++\rangle+\beta|---\rangle|-+\rangle \\ \overset{CNOT_{42}}{\mapsto}& \alpha|+++\rangle|++\rangle+\beta|---\rangle|++\rangle \\ \overset{CNOT_{52}}{\mapsto}& \alpha|+++\rangle|++\rangle+\beta|---\rangle|+-\rangle \\ \overset{CNOT_{53}}{\mapsto}& \alpha|+++\rangle|++\rangle+\beta|---\rangle|++\rangle. \end{aligned}$$

Now we have to make measurements M_1 and M_2. These are in Pauli Z basis, but we also have Hadamard in the circuit, thus effectively we measure in Pauli X bases. Which on qubits 4 and 5 will yield the outcomes $\{0,0\}$.

d) This was the case when no error occurred. Now we consider the case when a single phase-flip error, Pauli Z occurs. We have to repeat the same procedure as before, but now input $Z_i|\psi\rangle_L$, $i\in\{1,2,3\}$. Remember that $Z|pm\rangle=|mp\rangle$.

First Z_1:

$$Z_1(\alpha|+++\rangle+\beta|---\rangle)|++\rangle=(\alpha|-++\rangle+\beta|+--\rangle)|++\rangle.$$

$$\begin{aligned} CNOT_{41}(\alpha|-++\rangle+\beta|+--\rangle)|++\rangle) =& \alpha|-++\rangle|-+\rangle)+\beta|+--\rangle|++\rangle) \\ \overset{CNOT_{42}}{\mapsto}& \alpha|-++\rangle|-+\rangle)+\beta|+--\rangle|-+\rangle) \\ \overset{CNOT_{52}}{\mapsto}& \alpha|-++\rangle|-+\rangle)+\beta|+--\rangle|--\rangle) \\ \overset{CNOT_{53}}{\mapsto}& \alpha|-++\rangle|-+\rangle)+\beta|+--\rangle|-+\rangle). \end{aligned}$$

Thus the measurement outcome is $\{1,0\}$.

Next, Z_2:

$$Z_2(\alpha|+++\rangle+\beta|---\rangle)|++\rangle=(\alpha|+-+\rangle+\beta|-+-\rangle)|++\rangle.$$

Similarly as before one can calculate and show that the qubits 4 and 5 are in the state $|--\rangle$. Thus, the measurement gives $\{1,1\}$.

Finally, Z_3:

$$Z_3(\alpha|+++\rangle + \beta|---\rangle)|++\rangle = (\alpha|++-\rangle + \beta|--+\rangle)|++\rangle.$$

Similarly as before one can calculate and show that the qubits 4 and 5 are in the state $|-+\rangle$. Thus, the measurement gives $\{0,1\}$. We put this all together in the table:

Error	Syndrome		Corrections
$\mathbb{I}$	0	0	$\mathbb{I}$
Z_1	1	0	Z_1
Z_2	1	1	Z_2
Z_3	0	1	Z_3

e) Finally we have to add the correction operations in the Table. Measurement outcome indicates on which qubit the error occurred. For no error, we do not do corrections. For the rest see the table: